

Dirac theory on a space with linear Lie type fuzziness

Ahmad Shariati ¹
Mohammad Khorrami ²
Amir H. Fatollahi ³

Department of Physics, Alzahra University, Tehran 1993893973, Iran

Abstract

A spinor theory on a space with linear Lie type noncommutativity among spatial coordinates is presented. The model is based on the Fourier space corresponding to spatial coordinates, as this Fourier space is commutative. When the group is compact, the real space exhibits lattice characteristics (as the eigenvalues of space operators are discrete), and the similarity of such a *lattice* with ordinary lattices is manifested, among other things, in a phenomenon resembling the famous *fermion doubling* problem. A projection is introduced to make the dynamical number of spinors equal to that corresponding to the ordinary space. The actions for free and interacting spinors (with Fermi-like interactions) are presented. The Feynman rules are extracted and 1-loop corrections are investigated.

¹shariati@mailaps.org

²mamwad@mailaps.org

³fath@alzahra.ac.ir

1 Introduction

In recent years much attention has been paid to the formulation and study of field theories on noncommutative spaces. The motivation is partly the natural appearance of noncommutative spaces in some areas of physics, including recently in string theory. In particular it has been understood that the longitudinal directions of D-branes in the presence of a constant B -field background appear to be noncommutative, as seen by the ends of open strings [1–4]. In this case the coordinates satisfy the canonical relation

$$[\hat{x}_\mu, \hat{x}_\nu] = i \theta_{\mu\nu} \mathbf{1}, \quad (1)$$

in which θ is an antisymmetric constant tensor and $\mathbf{1}$ is the unit operator. The theoretical and phenomenological implications of such noncommutative coordinates have been extensively studied.

One direction to extend studies on noncommutative spaces is to consider spaces where the commutators of the coordinates are not constants. Examples of this kind are the noncommutative cylinder and the q -deformed plane (the Manin plane [5]), the so-called κ -Poincaré algebra [6] (see also [7–11]), and linear noncommutativity of the Lie algebra type [12] (see also [13, 14]). In the latter the dimensionless spatial position operators satisfy the commutation relations of a Lie algebra:

$$[\hat{x}_a, \hat{x}_b] = f^c{}_{ab} \hat{x}_c, \quad (2)$$

where $f^c{}_{ab}$'s are structure constants of a Lie algebra. One example of this kind is the algebra $SO(3)$, or $SU(2)$. A special case of this is the so called fuzzy sphere [15] (see also [16]), where an irreducible representation of the position operators is used which makes the Casimir of the algebra, $(\hat{x}_1)^2 + (\hat{x}_2)^2 + (\hat{x}_3)^2$, a multiple of the identity operator (a constant, hence the name sphere). One can consider the square root of this Casimir as the radius of the fuzzy sphere. This is, however, a noncommutative version of a two-dimensional space (sphere).

In [17–19] a model was introduced in which the representation was not restricted to an irreducible one, instead the whole group was employed. In particular the regular representation of the group was considered, which contains all representations. As a consequence in such models one is dealing with the whole space, rather than a sub-space, like the case of fuzzy sphere as a 2-dimensional surface. In [17] basic ingredients for calculus on a linear fuzzy space, as well as basic notions for a field theory on such a space, were introduced. In [18, 19] basic elements for calculating the matrix elements corresponding to transition between initial and final states, together with the explicit expressions for tree and one-loop amplitudes were given. It is observed that models based on Lie algebra type noncommutativity enjoy three features:

- They are free from any ultraviolet divergences if the group is compact.
- There is no momentum conservation in such theories.
- In the transition amplitudes only the so-called planar graphs contribute.

The reason for latter is that the non-planar graphs are proportional to δ -distributions whose dimensions are less than their analogues coming from the planar sector, and so their contributions vanish in the infinite-volume limit usually taken in transition amplitudes [19]. One consequence of a different mass-shell condition of these kinds of theory was explored in [20].

In [21] the classical mechanics defined on a space with $SU(2)$ fuzziness was studied. In particular, the Poisson structure induced by noncommutativity of $SU(2)$ type was investigated, for either the Cartesian or Euler parameterization of $SU(2)$ group. The consequences of $SU(2)$ -symmetry in such spaces on integrability, were also studied in [21]. In [22] the quantum mechanics on a space with $SU(2)$ fuzziness was examined. In particular, the commutation relations of the position and momentum operators corresponding to spaces with Lie-algebra noncommutativity in the configuration space, as well as the eigen-value problem for the $SU(2)$ -invariant systems were studied. The consequences of the Lie type noncommutativity of space on thermodynamical properties have been explored in [23, 24].

The purpose of this work is to develop a spinor theory on a space with linear Lie type noncommutativity (among spatial coordinates), specially corresponding to a Lie type noncommutativity of the form $SU(2)$, in which case the number of spatial coordinates is 3. The model is basically developed in the Fourier space, as it is commutative (contrary to the real space). When the group which corresponds to the noncommutativity is compact, the real space behaves in some sense like a lattice, as the eigenvalues of the coordinate operators are discrete. This is manifested, among other things, in the fact that the dynamical number of fermions is more than the corresponding number in the real space, similar to the famous *fermion doubling* problem arisen in fermion theories on ordinary lattices. A projection is introduced to make the dynamical number of spinors equal to the number corresponding to the ordinary space. The actions for free and Fermi-like interacting spinors are presented. As the momentum space is compact, the interacting theory is finite and does not suffer from ultraviolet divergences, contrary to the case of ordinary (commuting) space on which the theory is ultraviolet divergent, and not renormalizable. It is seen that for such theories, the 1-loop correction to the propagator has no non-planar contribution, contrary to the case of scalar fields on noncommutative spaces [19]. However, the 1-loop correction to the 4-point function is shown to get both planar and non-planar contributions.

The scheme of the rest of this paper is the following. In section 2, a brief introduction of the group algebra is given, mainly to fix notation. In section 3, the Dirac equation is presented in the momentum space, first in the case of the commutative space, then in the case of a noncommutative space. There it is shown that the number of dynamical degrees of spinors is more than the corresponding number in the ordinary space, and a projection is introduced to reduce the number of dynamical degrees of spinors to that of ordinary space. In section 4 the Dirac action on a noncommutative space is presented, for free fermions as well as fermions with Fermi-like interactions. The Feynman rules are extracted, and 1-loop corrections to the propagator and the 4-point function

are studied. Section 5 is devoted to the concluding remarks.

2 The group algebra

Assume that there exists a unique measure dU (up to a multiplicative constant) with the invariance properties

$$\begin{aligned} d(VU) &= dU, \\ d(UV) &= dU, \\ d(U^{-1}) &= dU, \end{aligned} \tag{3}$$

for any arbitrary element (V) of the group. For a compact group G , such a measure does exist. There are, however, groups which are not compact but for them as well such a measure exists. Examples are noncompact Abelian groups.

The meaning of (3), is that the measure is invariant under the left-translation, right-translation, and inversion. This measure, the (left-right-invariant) Haar measure, is unique up to a normalization constant, which defines the volume of the group:

$$\int_G dU = \text{vol}(G). \tag{4}$$

Using this measure, one constructs a vector space as follows. Corresponding to each group element U an element $\mathfrak{e}(U)$ is introduced, and the elements of the vector space are linear combinations of these elements:

$$f := \int dU f(U) \mathfrak{e}(U), \tag{5}$$

The group algebra is this vector space, equipped with the multiplication

$$f \bullet g := \int dU dV f(U) g(V) \mathfrak{e}(UV), \tag{6}$$

where (UV) is the usual product of the group elements. $f(U)$ and $g(U)$ belong to a field (here the field of complex numbers). It can be seen that if one takes the central extension of the group $U(1) \times \cdots \times U(1)$, the so-called Heisenberg group, with the algebra (1), the above definition results in the well-known star product of two functions, provided f and g are interpreted as the Fourier transforms of the functions.

So there is a correspondence between functionals defined on the group, and the group algebra. The definition (6) can be rewritten as

$$(f \bullet g)(W) = \int dV f(WV^{-1}) g(V). \tag{7}$$

The delta distribution is defined through

$$\int dU \delta(U) f(U) := f(\mathbf{1}), \tag{8}$$

where $\mathbf{1}$ is the identity element of the group.

Next, one can define an inner product on the group algebra. Defining

$$\langle \mathbf{e}(U), \mathbf{e}(V) \rangle := \delta(U^{-1} V), \quad (9)$$

and demanding that the inner product be linear with respect to its second argument and antilinear with respect to its first argument, one arrives at

$$\langle f, g \rangle = \int dU f^*(U) g(U). \quad (10)$$

Finally, one defines a star operation through

$$f^*(U) := f^*(U^{-1}). \quad (11)$$

This is in fact equivalent to definition of the star operation in the group algebra as

$$[\mathbf{e}(U)]^* := \mathbf{e}(U^{-1}). \quad (12)$$

It is then easy to see that

$$(f g)^* = g^* f^*, \quad (13)$$

$$\langle f, g \rangle = (f^* g)(\mathbf{1}). \quad (14)$$

3 The Dirac equation

To write the Dirac equation on a noncommutative space, let us begin with the Dirac equation on a commutative space in terms of the Fourier transform. For the 4 dimensional space-time, the following conventions are used,

$$\{\gamma^\sigma, \gamma^\rho\} = 2 \eta^{\sigma\rho}, \quad (15)$$

$$\gamma^0 = \mathbf{i} \beta, \quad (16)$$

$$\bar{\psi} = \psi^\dagger \beta, \quad (17)$$

where η is the Minkowski metric with the signature $(-, +, +, +)$.

3.1 The Dirac equation in the Fourier space

The Dirac equation in the Fourier space on a commutative space is

$$(\gamma^0 \partial_0 + \mathbf{i} \gamma^a k_a - \mu) \psi(t, U) = 0, \quad (18)$$

where

$$U := \exp(\ell k^a T_a), \quad (19)$$

$$k_a := \delta_{ab} k^b, \quad (20)$$

and

$$\mathrm{tr}(T_a T_b) = c \delta_{ab}, \quad (21)$$

$$\mathrm{tr}(T_a) = 0. \quad (22)$$

ℓ is a parameter of dimension length, and T_a 's are generators of some group, in some representation. But as long as (18) is studied, it is not important what the value of ℓ is, and what the group is, provided the dependence of the group element U on \mathbf{k} is one to one. If the latter condition is violated, then sending ℓ to zero effectively makes the dependence of U on \mathbf{k} one to one. It is then seen that the Dirac equation can be written as

$$\{\gamma^0 \partial_0 + \mathrm{i} c^{-1} \gamma^a \lim_{\ell \rightarrow 0} [\ell^{-1} \mathrm{tr}(T_a U)] - \mu\} \psi(t, U) = 0. \quad (23)$$

3.2 The Dirac equation on a noncommutative space

Equation (23) provides one with a way of writing the Dirac equation on a noncommutative space. This is done essentially by removing the limit $\ell \rightarrow 0$, and taking (as usual) T_a 's to be the generator of a Lie group G . So the equation reads

$$[\mathcal{D}(U)] \psi(t, U) = 0, \quad (24)$$

where the Dirac operator \mathcal{D} is defined as

$$\mathcal{D}(U) := \gamma^0 \partial_0 + \mathrm{i} c^{-1} \gamma^a \ell^{-1} \mathrm{tr}(T_a U) - \mu. \quad (25)$$

Using

$$\mathrm{tr}(T_a U) = \ell^{-1} \frac{\partial \mathrm{tr}(U)}{\partial k^a}, \quad (26)$$

one obtains

$$\mathcal{D}(U) = \gamma^0 \partial_0 + \mathrm{i} c^{-1} \gamma^a \ell^{-2} \frac{\partial \mathrm{tr}(U)}{\partial k^a} - \mu. \quad (27)$$

Of course c and $\mathrm{tr}(U)$ depend on the representation.

3.3 The Dirac equation for the gauge group $\mathrm{SU}(2)$

For the group $\mathrm{SU}(2)$, one has for the spin s representation

$$\mathrm{tr}(U) = \frac{\sin \left[\ell k \left(s + \frac{1}{2} \right) \right]}{\sin \frac{\ell k}{2}}, \quad (28)$$

where

$$k := \sqrt{\delta_{ab} k^a k^b}. \quad (29)$$

So,

$$\ell^{-1} \frac{\partial \mathrm{tr}(U)}{\partial k^a} = \frac{k_a}{k} \frac{s \sin[(s+1) \ell k] - (s+1) \sin(s \ell k)}{2 \sin^2(\ell k/2)}. \quad (30)$$

One also has

$$c = -\frac{s(s+1)(2s+1)}{3}. \quad (31)$$

So one obtains for the Dirac operator

$$\mathcal{D}(U) = \gamma^0 \partial_0 - \frac{3\{s \sin[(s+1)\ell k] - (s+1) \sin(s\ell k)\}}{s(s+1)(2s+1)[2\ell k \sin^2(\ell k/2)]} (i\gamma^a k_a) - \mu. \quad (32)$$

In the special case $s = 1/2$, this becomes

$$\mathcal{D}(U) = \gamma^0 \partial_0 + \frac{2 \sin(\ell k/2)}{\ell k} (i\gamma^a k_a) - \mu. \quad (33)$$

The mass shell condition can be obtained, similar to the case of commutative spaces, by multiplying the Dirac equation from the left by the conjugate operator. The result would be

$$\begin{aligned} 0 &= \left[\gamma^0 \partial_0 + \frac{2 \sin(\ell k/2)}{\ell k} (i\gamma^a k_a) + \mu \right] \\ &\quad \times \left[\gamma^0 \partial_0 + \frac{2 \sin(\ell k/2)}{\ell k} (i\gamma^a k_a) - \mu \right] \psi(t, U), \\ &= \left\{ -(\partial_0)^2 - \left[\frac{2 \sin(\ell k/2)}{\ell} \right]^2 - \mu^2 \right\} \psi(t, U), \end{aligned} \quad (34)$$

which results in the following mass shell condition

$$\omega^2 = \frac{2[1 - \cos(\ell k)]}{\ell^2} + \mu^2. \quad (35)$$

For the group $SU(2)$, the range of k to cover all of the group once, is

$$0 \leq (\ell k) \leq (2\pi). \quad (36)$$

So the energy is clearly not an increasing function of k . It is so for (ℓk) between 0 and π . It seems that the whole range of k produces two copies of the spinor field. That is similar to what arises in the context of spinor fields on regular lattices, the so-called *fermion doubling* problem [25]. There, corresponding to each direction there are two copies of the spinor field. So that corresponding to a four dimensional lattice there are 16 copies of the spinor field. One could get rid of the additional (redundant) fields, by introducing suitable projections which commute with the equation of motion operator, so that different eigenvectors of the projections satisfy the equation separately. This approach is similar to the momentum space formulation of the so-called *staggered fermions* in ordinary lattice gauge theories [25]. For the present case, one notices that changing (ℓk_a) to $[(\ell k_a) - (2\pi k_a/k)]$ is equivalent to changing U to $(-U)$. Such a transformation changes (ℓk) to $|\ell k - 2\pi|$, so it turns the region $(\ell k) \in [0, \pi]$ into $(\ell k) \in [\pi, 2\pi]$, and vice versa. So a possible projection could be constructed through the operator \mathcal{M} with

$$(\mathcal{M}\psi)(t, U) := M\psi(t, -U), \quad (37)$$

which results in

$$(\mathcal{M} \mathcal{D} \mathcal{M}^{-1})(U) := M [\mathcal{D}(-U)] M^{-1}. \quad (38)$$

Multiplying \mathcal{M} from left on (24), one arrives at

$$\{M [\mathcal{D}(-U)] M^{-1}\} (\mathcal{M} \psi)(t, U) = 0. \quad (39)$$

So $(\mathcal{M} \psi)$ satisfies the same equation ψ satisfies, provided

$$\{M [\mathcal{D}(-U)] M^{-1}\} = \mathcal{D}(U), \quad (40)$$

or

$$\begin{aligned} M \gamma^0 M^{-1} &= \gamma^0, \\ M \gamma^a M^{-1} &= -\gamma^a, \end{aligned} \quad (41)$$

which shows that M is proportional to β . Taking it to be the same as β , it is seen that

$$\mathcal{M}^2 = 1. \quad (42)$$

So the eigenvalues of \mathcal{M} are ± 1 . One can then decompose the spinor field as

$$\psi = \psi^+ + \psi^-, \quad (43)$$

where

$$\psi^\pm := \frac{1 \pm \mathcal{M}}{2} \psi. \quad (44)$$

Obviously the equations for ψ^+ and ψ^- are decoupled. So that one can take only one of the spinor fields, say ψ^- , and write the equation for that. This way there remains only one copy of the spinor field, and one effectively needs only the values of k corresponding to (ℓk) not larger than π , as $\psi^-(t, -U)$ is not independent of $\psi^-(t, U)$.

It is obvious that the above construction works for any group with the property that if U belongs to the group, $(-U)$ belongs to the group as well. Let's call such a group a double copy group. One also notes that \mathcal{M} is not the parity operator. The parity operator transforms k to $(-k)$, which is equivalent to transforming U to U^{-1} , while \mathcal{M} transforms U to $(-U)$.

4 The Dirac action on a noncommutative space

The Dirac equation (24) for a free fermion can be obtained from an action.

4.1 The Dirac action for a free field

The Dirac action for a free field on a noncommutative space is written as

$$S_{\text{free}} = \frac{1}{\sigma} \int dt \int dU \bar{\psi}(t, U^{-1}) [\mathcal{D}(U)] \psi(t, U), \quad (45)$$

where σ is a symmetry factor:

$$\sigma = \begin{cases} 1, & G \text{ is not a double copy group} \\ 2, & G \text{ is a double copy group} \end{cases}, \quad (46)$$

and the Haar measure dU is normalized so that

$$dU \sim \frac{d^D k}{(2\pi)^D}, \quad U \sim \mathbf{1}. \quad (47)$$

The symmetry factor ensures that in the limit ($\ell \rightarrow 0$), the commutative action is recovered with proper normalization.

Obviously, if G is a double copy group one has

$$\overline{(\mathcal{M}\psi)}(t, U) = [\bar{\psi}(t, -U)] M^{-1}, \quad (48)$$

which together with

$$d(-U) = dU, \quad (49)$$

shows that the action S_{free} enjoys the following symmetry

$$S_{\text{free}}(\mathcal{M}\psi) = S_{\text{free}}(\psi). \quad (50)$$

Using the action (45), the propagator (in the full Fourier space) is found to be

$$\check{\Delta}(\omega, U) = (i\hbar) [\check{\mathcal{D}}(\omega, U)]^{-1}, \quad (51)$$

where

$$\check{\mathcal{D}}(\omega, U) = -i\gamma^0 \omega + i c^{-1} \gamma^a \ell^{-1} \text{tr}(T_a U) - \mu. \quad (52)$$

so,

$$\begin{aligned} \check{\Delta}(\omega, U) &= (i\hbar) [-i\gamma^0 \omega + i c^{-1} \gamma^a \ell^{-1} \text{tr}(T_a U) - \mu]^{-1}, \\ &= \frac{i\hbar}{\omega^2 - (c\ell)^{-2} \text{tr}(T_b U) \text{tr}(T^b U) - \mu^2} \\ &\quad \times [-i\gamma^0 \omega + i c^{-1} \gamma^a \ell^{-1} \text{tr}(T_a U) + \mu]. \end{aligned} \quad (53)$$

For the group $\text{SU}(2)$, the action (45) is reduced to

$$\begin{aligned} S_{\text{free}} &= \frac{1}{2} \int dt \int dU \bar{\psi}(t, U^{-1}) [\mathcal{D}(U)] \psi(t, U), \\ &= \frac{1}{2} \int dt \int dU \bar{\psi}^-(t, U^{-1}) [\mathcal{D}(U)] \psi^-(t, U) \\ &\quad + \frac{1}{2} \int dt \int dU \bar{\psi}^+(t, U^{-1}) [\mathcal{D}(U)] \psi^+(t, U), \\ &=: S_{\text{free}}^- + S_{\text{free}}^+, \end{aligned} \quad (54)$$

where $\mathcal{D}(U)$ is of the form (33). The propagator would be

$$\begin{aligned}\check{\Delta}(\omega, U) &= (i\hbar) \left[-i\gamma^0 \omega + \frac{2 \sin(\ell k/2)}{\ell k} (i\gamma^a k_a) - \mu \right]^{-1}, \\ &= \frac{i\hbar}{\omega^2 - (4/\ell^2) \sin^2(\ell k/2) - \mu^2} \left[-i\gamma^0 \omega + \frac{2 \sin(\ell k/2)}{\ell k} (i\gamma^a k_a) + \mu \right].\end{aligned}\quad (55)$$

The action (54) contains two copies of the fermion field, as previously explained. So the proper action of a single free fermion field would be S_{free}^- (or S_{free}^+).

4.2 Interacting Dirac fields

An example for the interaction of Dirac fields is a Fermi like interaction, corresponding to the action

$$\begin{aligned}S_{\text{Fermi}} &= -\frac{g}{(j!)^2} \int dt \int dU_1 \cdots dU_{2j} \delta(U_1 \cdots U_{2j}) \\ &\quad \times [\bar{\psi}(t, U_1) \psi(t, U_2)] \cdots [\bar{\psi}(t, U_{2j-1}) \psi(t, U_{2j})].\end{aligned}\quad (56)$$

This interaction is not renormalizable in the ordinary space. But here there are no ultraviolet divergences (as far as the group is compact). Again, if G is a double copy group one could write actions which contain only one copy of the fermion field:

$$\begin{aligned}S_{\text{Fermi}}^- &= -\frac{g}{\sigma^{2j-1} (j!)^2} \int dt \int dU_1 \cdots dU_{2j} \delta(U_1 \cdots U_{2j}) \\ &\quad \times [\bar{\psi}^-(t, U_1) \psi^-(t, U_2)] \cdots [\bar{\psi}^-(t, U_{2j-1}) \psi^-(t, U_{2j})].\end{aligned}\quad (57)$$

The full action, would then be

$$\begin{aligned}S^- &= \frac{1}{\sigma} \int dt \int dU_1 dU_2 \delta(U_1 U_2) \bar{\psi}^-(t, U_1) [\mathcal{D}(U_2)] \psi^-(t, U_2) \\ &\quad - \frac{g}{\sigma^{2j-1} (j!)^2} \int dt \int dU_1 \cdots dU_{2j} \delta(U_1 \cdots U_{2j}) \\ &\quad \times [\bar{\psi}^-(t, U_1) \psi^-(t, U_2)] \cdots [\bar{\psi}^-(t, U_{2j-1}) \psi^-(t, U_{2j})].\end{aligned}\quad (58)$$

The vertex corresponding to such an interaction reads

$$\begin{aligned}\mathcal{V}_{\alpha_2 \alpha_4 \cdots \alpha_{2j}}^{\alpha_1 \alpha_3 \cdots \alpha_{2j-1}}(U_1, \dots, U_{2j}) &= \frac{1}{\sigma^{2j-1} (j!)^2} \frac{g}{i\hbar} 2\pi \delta(\omega_1 + \cdots + \omega_{2j}) \\ &\quad \times \sum_{\Pi, \Pi'} \zeta_{\Pi} \zeta_{\Pi'} \delta_{\alpha_2 \Pi'(1)}^{\alpha_2 \Pi(1)-1} \cdots \delta_{\alpha_2 \Pi'(j)}^{\alpha_2 \Pi(j)-1} \\ &\quad \times \delta(U_{2\Pi(1)-1} U_{2\Pi'(1)} \cdots U_{2\Pi(j)-1} U_{2\Pi'(j)}),\end{aligned}\quad (59)$$

where Π and Π' are j -permutations, and ζ_{Π} is the sign of the permutation Π (plus one for even permutations, and minus one for odd permutations).

For the 4-fermion interaction, the above would be

$$\mathcal{V}_{\alpha_2 \alpha_4}^{\alpha_1 \alpha_3}(U_1, \dots, U_4) = \frac{1}{2\sigma^3} \frac{g}{i\hbar} 2\pi \delta(\omega_1 + \dots + \omega_4) \\ \times [\delta_{\alpha_2}^{\alpha_1} \delta_{\alpha_4}^{\alpha_3} \delta(U_1 U_2 U_3 U_4) - \delta_{\alpha_4}^{\alpha_1} \delta_{\alpha_2}^{\alpha_3} \delta(U_1 U_4 U_3 U_2)], \quad (60)$$

where use has been made of the fact that

$$\delta(U U') = \delta(U' U). \quad (61)$$

and so forth. It is seen that the above vertex respects a deformed momentum conservation. In commutative space, the momentum conservation is that the sum of all momenta should vanish. In noncommutative space, however, processes are allowed for which the product of group elements are unit, and as different orderings in the products are possible, there could be several conservation delta functions in the vertex, which are not the same. In the above, for example, there are two different delta functions. A similar thing occurs in theories defined on κ -deformed spaces. In these theories, the ordinary summation of momenta in each vertex is replaced by a new rule of summation, occasionally called as doted-sum ($\dot{+}$) [10]. This new sum, in contrast to the ordinary sum, is non-Abelian, and as a consequence, the delta functions's corresponding to different possible orderings of legs are different [10, 11].

4.3 1-loop correction of the 2-point function

The 2-point function has two external legs 1 and 2. The 1-loop correction is simply the vertex-function (60), contracted with the propagator corresponding to the legs 3 and 4:

$$\Gamma_1^{(2)\alpha_1}_{\alpha_2} = \int dU_3 dU_4 \frac{d\omega_3}{2\pi} \frac{d\omega_4}{2\pi} \delta(U_3 U_4) (2\pi) \delta(\omega_3 + \omega_4) \mathcal{V}_{\alpha_2 \alpha_4}^{\alpha_1 \alpha_3} \check{\Delta}^{\alpha_4}_{\alpha_3}(\omega_3, U_3) \\ = \frac{g}{i\hbar} \frac{1}{2\sigma^3} (2\pi) \delta(\omega_1 + \omega_2) \delta(U_1 U_2) \int dU_3 \frac{d\omega_3}{2\pi} \frac{i\hbar}{\omega_3^2 + O(U_3)} \\ \times \{\delta_{\alpha_2}^{\alpha_1} \text{tr}[\check{\mathcal{D}}'(\omega_3, U_3)] - \check{\mathcal{D}}'^{\alpha_1}_{\alpha_2}(\omega_3, U_3)\}, \quad (62)$$

where

$$\check{\mathcal{D}}'(\omega, U) := -i\gamma^0 \omega + i c^{-1} \gamma^a \ell^{-1} \text{tr}(T_a U) + \mu, \\ O(U) := -(c\ell)^{-2} \text{tr}(T_b U) \text{tr}(T^b U) - \mu^2. \quad (63)$$

The coefficient $\delta(U_1 U_2)$ shows that for the propagator, up to 1-loop correction the analog of momentum conservation still holds.

The above expression for 2-point function may be contrasted with the similar one for the scalar fields [19]. In the case for scalars, there is a term in which the δ -function consists the loop variable, and so could not be brought out the integral. In that case, such a term is called as the non-planar contribution. Here, for spinors, we see that such a term is absent, and the contribution is totally planar.

4.4 1-loop correction of the 4-point function

The aim of this subsection is to investigate the possibility of non-planar contributions. Labeling the external legs 1 through 4, it is seen that there are 6 distinct combinations of pseudo-conservation terms:

$$\begin{aligned}
\mathcal{C}_I &= \delta(U_1 U_2 U_5 U_6) \delta(U_3 U_4 U_6^{-1} U_5^{-1}), \\
\mathcal{C}_{II} &= \delta(U_1 U_2 U_5 U_6) \delta(U_3 U_5^{-1} U_6^{-1} U_4), \\
\mathcal{C}_{III} &= \delta(U_1 U_4 U_5 U_6) \delta(U_3 U_2 U_6^{-1} U_5^{-1}), \\
\mathcal{C}_{IV} &= \delta(U_1 U_4 U_5 U_6) \delta(U_3 U_5^{-1} U_6^{-1} U_2), \\
\mathcal{C}_V &= \delta(U_1 U_6 U_3 U_5^{-1}) \delta(U_5 U_2 U_6^{-1} U_4), \\
\mathcal{C}_{VI} &= \delta(U_1 U_6 U_3 U_5^{-1}) \delta(U_5 U_4 U_6^{-1} U_2),
\end{aligned} \tag{64}$$

where the labels 5 and 6 refer to internal legs. Integrating over U_6 , the corresponding contributions become

$$\begin{aligned}
\mathcal{C}'_I &= \delta(U_1 U_2 U_3 U_4), \\
\mathcal{C}'_{II} &= \delta(U_1 U_2 U_5 U_4 U_3 U_5^{-1}), \\
\mathcal{C}'_{III} &= \delta(U_1 U_4 U_3 U_2), \\
\mathcal{C}'_{IV} &= \delta(U_1 U_4 U_5 U_2 U_3 U_5^{-1}), \\
\mathcal{C}'_V &= \delta(U_1 U_4 U_5 U_2 U_3 U_5^{-1}), \\
\mathcal{C}'_{VI} &= \delta(U_1 U_2 U_5 U_4 U_3 U_5^{-1}).
\end{aligned} \tag{65}$$

It is seen that of these six channels, the first and the third correspond to planar contributions, while others correspond to non-planar ones.

5 Concluding remarks

A spinor theory on a space with linear Lie type noncommutativity among spatial coordinates was presented. It was shown that the dynamical number of spinors could be more than the number corresponding to the commutative spaces, as a result of the lattice-like nature of the noncommutative space. This is similar to the famous *fermion doubling* problem arisen in the case of regular lattices. A projection was introduced to remove the additional degrees of freedom. Actions for free and Fermi-like interacting spinors were presented, and were specialized to the case where the group corresponding to the noncommutativity is $SU(2)$. The Feynman rules were extracted and 1-loop corrections to the 2- and 4-point functions were studied. It was shown that up to 1-loop, there is no non-planar contribution in the 2-point function, while there are planar as well as non-planar contributions in the 4-point function.

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References

- [1] N. Seiberg & E. Witten, JHEP **9909** (1999) 032.
- [2] A. Connes, M. R. Douglas, & A. Schwarz, JHEP **9802** (1998) 003.
- [3] M. R. Douglas & C. Hull, JHEP **9802** (1998) 008.
- [4] H. Arfaei & M. M. Sheikh-Jabbari, Nucl. Phys. **B526** (1998) 278.
- [5] Yu. I. Manin, “Topics in Noncommutative Geometry”, (Princeton University Press, 1991).
- [6] J. Lukierski, H. Ruegg, A. Nowicki & V. N. Tolstoy, Phys. Lett. **B264** (1991) 331.
- [7] M. Chaichian, A. Demichev, & P. Presnajder, Nucl. Phys. **B567** (2000) 360;
J. Math. Phys. **41** (2000) 1647.
- [8] S. Majid & H. Ruegg, Phys. Lett. **B334** (1994) 348.
- [9] J. Lukierski, H. Ruegg, & W. J. Zakrzewski, Annals Phys. **243** (1995) 90;
J. Lukierski & H. Ruegg, Phys. Lett. **B329** (1994) 189;
G. Amelino-Camelia, Phys. Lett. **B392** (1997) 283.
- [10] G. Amelino-Camelia & M. Arzano, Phys. Rev. **D65** (2002) 084044;
G. Amelino-Camelia, M. Arzano, & L. Doplicher, in “25th Johns Hopkins Workshop on Current Problems in Particle Theory”, hep-th/0205047.
- [11] P. Kosinski, J. Lukierski, & P. Maslanka, Phys. Rev. **D62** (2000) 025004;
D. Robbins & S. Sethi, JHEP **07** (2003) 034;
H. Grosse & M. Wohlgenannt, Nucl. Phys. **B748** (2006) 473.
- [12] H. S. Snyder, Phys. Rev. **71** (1947) 38.
- [13] J. Madore, S. Schraml, P. Schupp, & J. Wess, Eur. Phys. J. **C16** (2000) 161.
- [14] N. Sasakura, JHEP **0005** (2000) 015;
S. Imai & N. Sasakura, JHEP **0009** (2000) 032;
Y. Sasai & N. Sasakura, 0711.3059 [hep-th].
- [15] J. Madore, Class. Quant. Grav. **9** (1992) 69.
- [16] P. Presnajder, Mod. Phys. Lett. **A18** (2003) 2431;
H. Grosse & P. Presnajder, Lett. Math. Phys. **46** (1998) 61;
Lett. Math. Phys. **33** (1995) 171.
- [17] A. H. Fatollahi & M. Khorrami, Europhys. Lett. **80** (2007) 20003.

- [18] H. Komaie-Moghaddam, A. H. Fatollahi, & M. Khorrami, Eur. Phys. J. **C53** (2008) 679.
- [19] H. Komaie-Moghaddam, M. Khorrami, & A. H. Fatollahi, Phys. Lett. **B661** (2008) 226.
- [20] A. Shariati, M. Khorrami, & A. H. Fatollahi, Europhys. Lett. **81** (2008) 40003.
- [21] M. Khorrami, A. H. Fatollahi, & A. Shariati, J. Math. Phys. **50** (2009) 072902.
- [22] A. H. Fatollahi, A. Shariati, & M. Khorrami, Eur. Phys. J. **C60** (2009) 489.
- [23] H. Shin & K. Yoshida, Nucl. Phys. **B701** (2004) 380;
W.-H. Huang, JHEP **0908** (2009) 102.
- [24] A. Shariati, M. Khorrami, & A. H. Fatollahi, J. Phys. A: Math. Theor. **43** (2010) 285001.
- [25] H. J. Rothe, “Lattice Gauge Theories: An Introduction”, 3rd ed. , (World Scientific, 2005).